3.6 Lecture 11: Implicit differentiation, and specific antiderivatives

3.6.1 The remaining inverse trigonometric functions

We don’t tend to worry about the rest, because they can be expressed in terms of the ones we know. For example,

\[ y = \arccsc(x) \Rightarrow x = \csc(y) = \frac{1}{\sin(y)} \Rightarrow \sin(y) = \frac{1}{x} \Rightarrow y = \arcsin(1/x) \]

so that

\[ \arccsc(x) = \arcsin(1/x). \]

Similarly,

\[ \arccsc(x) = \arcsin(1/x) \]

and

\[ \arccot(x) = \arctan(1/x). \]

They’re interesting, but since most calculators don’t even have these functions as buttons, it’s kind of pointless to use them.

3.6.2 Application of the chain rule: Implicit differentiation

A function \( y = f(x) \) is a particular kind of relation between the variables \( x \) and \( y \) — one whose graph passes the vertical line test. If our variables satisfy a relation like

\[ x^2 + y^2 = 9 \]

then the corresponding graph is not a function, and does not pass the vertical line test. However, we know we can decompose this graph into pieces, such that each piece is a function; in this case, the graph is the union of the graphs of

\[ y = \sqrt{9 - x^2} \quad \text{and} \quad y = -\sqrt{9 - x^2}. \]

Now here’s the clever idea: if we want to find the slope of the tangent line to the circle at a certain point, do we really need to solve for \( y \) in terms of \( x \)? After all, if we know that \( y \) is a function of \( x \) near each point, then we can differentiate \( y \) with respect to \( x \). For example,

\[ \frac{d}{dx}(y^2) = 2y \frac{dy}{dx}. \]

What this implies is that sometimes we can solve for \( y' \) without first having to solve for \( y \).

**Remark 3.60.** We used this idea when using logarithmic differentiation: that we can apply the chain rule to a function on \( y \) (like \( \ln(y) \)).

For example, if \( x^2 + y^2 = 9 \) then near any point we can think of both sides as being functions of \( x \); since they’re equal, their derivatives are equal. So we have

\[ 2x + 2yy' = 0 \]

and we can solve for

\[ y' = -\frac{x}{y}. \]

We check that at various points \((x, y)\) on the circle, this formula does indeed give the correct slope of the tangent line.
Remark 3.61. Notice that our answer for the derivative in this case contains both \( x \) and \( y \)! That’s because unlike a function, where \( x \) is enough to determine the point on the graph, here we’ll need both coordinates because there might be more than one point with that \( x \)-coordinate.

So:

- At a theoretical level, implicit differentiation is saying that if near a certain point \( y \) is a differentiable function of \( x \), then the derivative exists even if we can’t actually find a formula for \( y \) in terms of \( x \). As a consequence of the chain rule, you can therefore differentiate the relation itself to deduce a formula for \( y' \).

- At a mechanical level, implicit differentiation is saying that if you have an equation with variables which depend on \( x \), then you can differentiate both sides with respect to \( x \) using the chain rule (remembering that \( \frac{dx}{dx} = 1 \), but \( \frac{dy}{dx} = y' \), for example).

Example 3.62. Find the equation of the tangent line to the curve

\[ x^{2/3} + y^{2/3} = 5 \]

at the point \((1, 8)\).

Solution: we sketch the curve (called an astroid) and note that \((1, 8)\) is indeed a point on the curve. We differentiate both sides with respect to \( x \):

\[
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0
\]

whence

\[
y' = -\frac{x^{-1/3}}{y^{-1/3}} = \left(\frac{-y}{x}\right)^{1/3}.
\]

At the point \((1, 8)\), we see \(y' = (-8/1)^{1/3} = -2\). Therefore the equation for the tangent line is

\[
y - 8 = -2(x - 1) \quad \text{or} \quad y = -2x + 10
\]

which looks about right from the graph.

Example 3.63. The bifolium has equation

\[(x^2 + y^2)^2 = 4xy^2.\]

Three points on the graph are \((0, 0)\), \((1, 1)\) and \((3/4, \sqrt{3}/4)\). Find the slope of the tangent line, when defined.

Solution: We differentiation both sides with respect to \( x \) to get

\[
2(x^2 + y^2)(2x + 2yy') = 4y^2 + 4x(2yy').
\]

Now we isolate and solve for \(y'\):

\[
4x(x^2 + y^2) + 4y(x^2 + y^2)y' = 4y^2 + 8xyy'
\]

whence

\[
(4y(x^2 + y^2) - 8xy)y' = 4y^2 - 4x(x^2 + y^2),
\]

or

\[
y' = \frac{y^2 - x(x^2 + y^2)}{y(x^2 + y^2) - 2xy}.
\]

At \((0, 0)\), this is \(0/0\) so undefined. On the graph we see that there’s a huge mess at the origin; of course there’s no tangent line.
At \((1, 1)\) this is \(-2/0\) so again undefined; but on the graph we see that in fact there’s a vertical tangent line at this point. (So it’s not a function of \(x\) there; rather, \(x\) is a function of \(y\).)

At \((3/4, \sqrt{3}/4)\), we have
\[
y' = \frac{3/16 - (3/4)(3/4)}{(\sqrt{3}/4)(3/4) - (3\sqrt{3}/8)} = \frac{2\sqrt{3}}{3}
\]
which looks reasonable from the graph.

You can even find the second derivative this way.

**Example 3.64.** Suppose \(\sqrt{x} + \sqrt{y} = 1\). Find \(y'\) and \(y''\) at the point \((1/4, 1/4)\).

We have
\[
\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0
\]
whence
\[
y' = -\frac{\sqrt{y}}{\sqrt{x}}.
\]
So at that point the slope is \(-1\). The second derivative is
\[
y'' = \frac{-\sqrt{x} y' - \sqrt{y} y'}{2\sqrt{y} x}\n\]
\[
= \frac{1}{x} \left( \frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}} \right)
\]
after simplifying with \(y' = -\frac{\sqrt{y}}{\sqrt{x}}\). Therefore at that point the concavity is \(y'' = 4\); it’s concave up. This makes sense from the graph.

Do look at the examples in the book, which come with nice pictures of these strange algebraic curves.

### 3.6.3 Practice with antiderivatives

See [S, Section 4.8].

### 3.6.4 Quick summary of integrals we now know

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \quad \text{if } n \neq -1
\]
\[
\int x^{-1} \, dx = \ln |x| + c
\]
\[
\int a^x \, dx = \frac{1}{\ln(a)} a^x + c \quad \text{for any constant } a > 0
\]
\[
\int \sin(x) \, dx = -\cos(x) + c
\]
\[
\int \cos(x) \, dx = \sin(x) + c
\]
\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + c
\]
\[
\int \frac{1}{1+x^2} \, dx = \arctan(x) + c
\]
and we also know the anti-derivatives of \(\sec^2(x), \sec(x)\tan(x), \csc^2(x)\) and \(\csc(x)\cot(x)\).

This is our basic vocabulary of anti-derivatives, the ones that we know at a glance. After the fall break, we’ll tackle anti-product rule and anti-chain rule and other methods which will let us (sometimes) find antiderivatives of functions which are composed of these basic building blocks.
3.6.5 Finding particular anti-derivatives

We know that if $F(x)$ is an anti-derivative of $f(x)$, then so is

$$F(x) + c$$

for any constant $c$. For the purposes of calculating a definite integral, we don’t care about the ‘$+c$’ since it cancels out when you evaluate $F(b) - F(a)$. But what if you’re looking for a particular anti-derivative?

**Example 3.65.** Find $f$ if $f'(x) = x^3 - \frac{3}{\sqrt{1-x^2}}$ and $f(0) = 4$.

**Solution:** We start by find the general anti-derivative (that is, the indefinite integral) of

$$f'(x) = \sin(x) + x^3 - \frac{3}{\sqrt{1-x^2}}$$

which is

$$f(x) = -\cos(x) + \frac{1}{4}x^4 - 3\arcsin(x) + c,$$

for an arbitrary constant $c$. We can check that indeed $f'(x)$ is what we said it was.

Now consider that we need to satisfy $f(0) = 4$. This gives the equation

$$-\cos(0) + \frac{14}{0} - 3\arcsin(0) + c = 4$$

which simplifies to

$$-1 + c = 4 \quad \Rightarrow \quad c = 5.$$

Therefore there is exactly one answer:

$$f(x) = -\cos(x) + \frac{1}{4}x^4 - 3\arcsin(x) + 5.$$  

**Example 3.66.** Find $f$ if $f''(x) = \sqrt{x+1}$, $f(0) = 5$ and $f(1) = 0$.

**Solution:** Recall that $f''$ is the derivative of $f'$. So if

$$f''(x) = \frac{\sqrt{x+1}}{x} = x^{-1/2} + x^{-2/3}$$

then

$$f'(x) = 2x^{1/2} + 3x^{1/3} + c$$

for some arbitrary constant $c$. Therefore

$$f(x) = \frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + cx + d$$

for some OTHER arbitrary constant $d$.

Now $f(0) = d$ so $d = 5$; and $f(1) = \frac{4}{3} + \frac{9}{4} + c + 5$ so

$$\frac{4}{3} + \frac{9}{4} + c + 5 = 0 \quad \Rightarrow \quad c = 5 - \frac{9}{4} - \frac{4}{3} = \frac{60 - 27 - 16}{12} = \frac{17}{12}. $$

Therefore the only function $f$ satisfying all the given conditions is

$$f(x) = \frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + \frac{17}{12}x + 5.$$  

Recall that the derivative of the position function is the velocity function; and the derivative of velocity is acceleration.
Example 3.67. A particle moves along a rectilinear path with acceleration profile

\[ a(t) = 5 \cos(x) + e^{x+3} + 2 \]

in m/s². Its initial velocity (at \( t = 0 \)) was \( v_0 = 12 \text{m/s} \) and its initial position was \( s_0 = 0 \text{m} \). Find \( s(t) \).

Solution: If

\[ a(t) = 5 \cos(x) + e^{x+3} + 2 \]

then

\[ v(t) = 5 \sin(x) + e^{x+3} + 2x + c \]

for some value \( c \). But at \( t = 0 \) we must have \( v(0) = v_0 = 12 \) so

\[ 5 \sin(0) + e^3 + 2(0) + c = 12 \implies c = 12 - e^3. \]

Therefore the velocity function is

\[ v(t) = 5 \sin(x) + e^{x+3} + 2x + 12 - e^3 \text{ m/s}. \]

Now the anti-derivative gives us the position function:

\[ s(t) = -5 \cos(x) + e^{x+3} + x^2 + 12x - e^3x + d \]

for some constant \( d \). But \( s(0) = 0 \) implies

\[ -5 + e^3 + d = 0 \implies d = 5 - e^3. \]

Therefore the particle had position function

\[ s(t) = -5 \cos(x) + e^{x+3} + x^2 + (12 - e^3)x + (5 - e^3) \text{ m}. \]

When the domain of a function is a union of disjoint intervals (like \( \frac{1}{2}, \text{ or } \sec(x) \)), its most general anti-derivative allows for the possibility of different vertical shifts on each subinterval.

Example 3.68. Find a function \( f \) such that \( f'(x) = \sec^2(x) \), \( f(0) = 1 \) and \( f(\pi) = 2 \).

Solution: We start by saying \( f(x) = \tan(x) + c \). Then \( f(0) = 1 \) implies \( c = 1 \). But \( f(x) = \tan(x) + 1 \) doesn’t satisfy \( f(\pi) = 2 \); we should have chosen \( c = 2 \). What’s going on?

We sketch the graph of \( \tan(x) \) and see that 0 and \( \pi \) are in different connected components of the domain. So our solution could be

\[ f(x) = \begin{cases} 
\tan(x) + 1 & \text{if } -\pi/2 < x < \pi/2 \\
\tan(x) + 2 & \text{if } \pi/2 < x < 3\pi/2 \\
\tan(x) & \text{for all other } x, x \neq k\pi/2 \text{ for an odd integer } k
\end{cases} \]

Notice that the last line was arbitrary; we could have chosen any vertical shifts of \( \tan(x) \) in the other regions of the domain. In any case, we see that we have produced at least one function with all the desired attributes.