3.7 Lecture 12: The substitution method for integration

So far we can only calculate indefinite integrals when the integrand is a function whose anti-derivative we already know. This is a fairly small list (see [S, Table 1 in Chapter 5.3], for example).

Today we’ll learn and practice a method which is based on undoing the chain rule; we call it the method of substitution.

Recall that the chain rule tells us that

$$\int f'(g(x))g'(x)dx = f(g(x)) + c.$$ But how would we recognize that our integrand has this special form? This is where the special notation of integrals comes in handy.

Set

$$u = g(x)$$

then what we’ll write is:

$$\frac{du}{dx} = g'(x) \Rightarrow du = g'(x) \, dx.$$ This is just notation — a clever way to keep track of things. For now the integral is

$$\int f'(g(x))g'(x)dx = \int f'(u)du = f(u) + c = f(g(x)) + c.$$ Let’s do some examples.

**Example 3.69.** Find

$$\int xe^{x^2} \, dx.$$ The trickiest part of this integral is $e^{x^2}$, which is a composition of two functions. We try substituting for the innermost function:

$$u = x^2$$

so

$$du = 2x \, dx \Rightarrow x \, dx = \frac{1}{2} du$$

Now we have to rewrite the integral in terms of $u$:

$$\int xe^{x^2} \, dx = \int (e^{x^2})x \, dx$$

$$= \int e^u \frac{1}{2} du$$

$$= \frac{1}{2} \int e^u du$$

$$= \frac{1}{2} e^u + c$$

$$= \frac{1}{2} e^{x^2} + c.$$ Check by differentiating: yes!

Notice that we converted the integral to one of the variable $u$, and then at the end reverted back to $x.$
Example 3.70. Find
\[ \int \frac{\sec^2(1/x)}{x^2} \, dx. \]
Again, the trickiest part is \( \sec^2(1/x) \) and we know that \( \sec^2(u) \) has a nice anti-derivative; so we try
\[ u = 1/x \Rightarrow du = -\frac{1}{x^2} \, dx \Rightarrow \frac{1}{x^2} \, dx = -du \]
which gives
\[
\int \frac{\sec^2(1/x)}{x^2} \, dx = \int \sec^2(1/x) \frac{1}{x^2} \, dx \\
= \int \sec^2(u)(-1) \, du \\
= - \int \sec^2(u) \, du \\
= - \tan(u) + c \\
= - \tan(1/x) + c.
\]
Check by differentiating: yes!

3.7.1 Other situations where you might try substitution

Example 3.71. Find
\[ \int \frac{e^x}{e^x + 1} \, dx. \]
There isn’t an obvious composition of functions in this case. But the thing which is making this integral difficult is that there is a sum in the denominator, and we notice that the derivative of \( u = e^x + 1 \) is \( du = e^x \, dx \), which is right there in the numerator.
So we get
\[
\int \frac{1}{e^x + 1} e^x \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |e^x + 1| + c = \ln(e^x + 1) + c.
\]
Substitution is a great thing to try whenever you see both a function and its derivative in the integrand.

Example 3.72. Find
\[ \int \frac{\arctan(x)}{1 + x^2} \, dx. \]
This time, there is no composition of functions, so it’s not obvious that substitution is the thing to do. But we notice that the integrand is a product of a function \( u = \arctan(x) \) and its derivative \( du = \frac{1}{1+x^2} \, dx \), so we will just go ahead and make the substitution and see what happens:
\[
\int \frac{\arctan(x)}{1 + x^2} \, dx = \int u \, du = \frac{1}{2} u^2 + c = \frac{1}{2} (\arctan(x))^2 + c.
\]
We check by differentiation.
So what happened here was: we couldn’t see the term \( f'(g(x)) \) that normally clues us in because in this case \( f(u) = \frac{1}{2} u^2 \) and so \( f'(u) = u \).

Be willing to try a substitution, to see if it can work out.
3.7.2 What about definite integrals?

FIRST METHOD:

Example 3.73. Find
\[ \int_0^1 (3 - 2t)^5 \, dt \]

We could just multiply this out, but that’s a waste: this is a composition of functions, and we could just do the substitution
\[ u = 3 - 2t \quad \Rightarrow \quad du = -2 \, dt \quad \Rightarrow \quad dt = -\frac{1}{2} \, du \]

This time, though, we have a definite integral. The limits mean that \( t \) runs from 0 to 1 — so what are the limits on \( u \)? Use the formula:
\[ t = 0 \Rightarrow u = 3 - 2t = 3, \quad t = 1 \Rightarrow u = 3 - 2t = 1 \]

Therefore, we have
\[ \int_0^1 (3 - 2t)^5 \, dt = \int_3^1 u^5 \left( -\frac{1}{2} \right) \, du. \]

NOTICE that we have been EXTREMELY careful to make a perfect translation from \( t \) to \( u \): yes, the limits go “backwards” but that’s what the substitution did, and you can’t just swap it because you don’t like it.

Anyway: we thus have
\[
\begin{align*}
\int_0^1 (3 - 2t)^5 \, dt &= \int_3^1 u^5 \left( -\frac{1}{2} \right) \, du \\
&= -\frac{1}{2} \int_3^1 u^5 \, du \\
&= -\frac{1}{2} \left[ \frac{1}{6} u^6 \right]_3^1 \\
&= -\frac{1}{2} \cdot \frac{1}{6} (1^6 - 3^6) \\
&= -\frac{1}{12} (1 - 243) = \frac{121}{6}
\end{align*}
\]

and we notice that the answer is positive, which is what we totally expected.

When you use substitution on a definite integral, change the limits of integration, too!!

SECOND METHOD:

Alternatively: remove the limits of integration, solve the indefinite integral, and then put the limits back to evaluate the definite integral.

Example 3.74. Find
\[ \int_1^2 \frac{\sin(\ln(3x))}{x} \, dx. \]

This is a definite integral, but as a first step let’s find
\[ \int \frac{\sin(\ln(3x))}{x} \, dx. \]
The integrand is a composition of functions (in fact, of three functions). We could try \( u = 3x \), or \( u = \ln(3x) \). Let’s choose this second one:

\[
u = \ln(3x) \quad \Rightarrow \quad du = \frac{1}{3x} \cdot 3 \, dx = \frac{1}{x} \, dx
\]

which is wonderful, since we have \( \frac{1}{x} \, dx \) in our integrand. So we go ahead and make the substitution:

\[
\int \frac{\sin(\ln(3x))}{x} \, dx = \int \sin(u) \, du = -\cos(u) + c = -\cos(\ln(3x)) + c.
\]

We check that this is correct by differentiation.

Now we go back and solve the definite integral. Remember, for the definite integral we can pick any one anti-derivative, like \( -\cos(\ln(3x)) \):

\[
\int_1^2 \frac{\sin(\ln(3x))}{x} \, dx = -\cos(\ln(3)) - (-\cos(\ln(6))) = -\cos(\ln(3)) + \cos(\ln(6))
\]

(and, remembering that \( \ln(6) \) and \( \ln(3) \) represent quantities in radians, you can give a decimal approximation on your calculator).

Note that the advantage of this second method is that (a) you can check your anti-derivative is correct and (b) you don’t have to convert your limits of integration, because you only evaluate the limits when you’re back in terms of \( x \).

The disadvantage is that it is longer to do.

### 3.7.3 Even and odd functions

Substitution gives us an easy proof of a nice property we’ve known for a while.

**Example 3.75.** Suppose \( f(x) \) is an odd function. Prove that \( \int_{-a}^{a} f(x) \, dx = 0 \) for any real number \( a \).

**Proof:** We have

\[
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx
\]

Now let’s make the substitution \( u = -x \) in the first integral on the right side. This gives

\[
\int_{-a}^{0} f(x) \, dx = \int_{a}^{0} f(-u)(-1) \, du = -\int_{a}^{0} f(-u) \, du = \int_{0}^{a} f(-u) \, du
\]

and since \( f(-u) = -f(u) \) (since it’s an odd function), we have

\[
\int_{-a}^{0} f(x) \, dx = -\int_{0}^{a} f(u) \, du.
\]

But remember that in a definite integral, the name of the letter doesn’t matter; we have

\[
-\int_{0}^{a} f(u) \, du = -\int_{0}^{a} f(x) \, dx
\]

so in fact the sum above cancels out and our final answer is zero, as required. (We didn’t do a substitution here for the last step: we just interpreted the integral as telling us the (net) area under the curve between 0 and \( a \).)

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4although you could also have tried \( u = 3x \) first, which works fine but yields \( \frac{1}{3} \int \sin(\ln(u)) \, du \), and you have to do a second substitution \( v = \ln(u) \). So it all works out just fine, it just takes a little longer.
Example 3.76. Find
\[ \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin(x)}{1 + x^6} \, dx. \]
Well, this integral would be hopeless but for the fact that the integrand is an odd function and the interval is symmetric about the origin, so it’s zero.

By a very similar argument, we can prove that if \( f \) is even (meaning \( f(x) = f(-x) \) for all \( x \)), then
\[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx. \]

3.7.4 Trying a substitution in the hopes of simplifying a complicated integrand

Sometimes, you try a substitution without actually realizing what \( f(u) \) is going to be.

Example 3.77. Find
\[ \int \frac{x^3}{\sqrt{x^2 + 4}} \, dx. \]
We see that the toughest part is \( \sqrt{x^2 + 4} \). We have a few choices (namely \( u = x^2 \) or \( u = x^2 + 4 \)); let’s try \( u = x^2 + 4 \).

So \( u = x^2 + 4 \), \( \Rightarrow \) \( du = 2x \, dx \) \( \Rightarrow \) \( x \, dx = \frac{1}{2} du \)

So now we try to rewrite our integral in terms of \( u \):
\[
\int \frac{x^3}{\sqrt{x^2 + 4}} \, dx = \int \frac{1}{\sqrt{x^2 + 4}} x^2 \cdot x \, dx \\
\neq \int \frac{1}{\sqrt{u}} (x^2)^{1/2} \, du \quad \text{WRONG! you can’t mix } x \text{ and } u \\
= \int \frac{1}{\sqrt{u}} (u - 4)^{1/2} \, du \quad \text{SUCCESS!} \\
= \frac{1}{2} \int \frac{u - 4}{u^{1/2}} \, du \\
= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) \, du \\
= \frac{1}{2} \left( \frac{1}{3/2} u^{3/2} - \frac{4}{1/2} u^{1/2} \right) + c \\
= \frac{2}{3} (x^2 + 4)^{3/2} - 8(x^2 + 4)^{1/2} + c \\
= \frac{1}{3} (x^2 + 4)^{3/2} - 4\sqrt{x^2 + 4} + c
\]

and again **we check by differentiating** — this time, even the check is a lot more work!

Lesson: sometimes you just try a substitution to see if it will work out.
But substitution isn’t only for cases where there’s an obvious composition of functions.

Example 3.78. Find
\[ \int \frac{1}{x^{1/3} + 1} \, dx \]
Well, this one seems hopeless but the nasty part is \( x^{1/3} \) so let’s just try
\[ u = x^{1/3} \quad \Rightarrow \quad du = \frac{1}{3} x^{-2/3} \, dx. \]

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We certainly don’t have $\frac{1}{3}x^{-2/3} \, dx$ in our integrand; but since $u = x^{1/3}$, it follows that $x^{-2/3} = u^{-2}$. Therefore we can rewrite:

$$du = \frac{1}{3}x^{-2/3} \, dx \Rightarrow \frac{1}{3}u^{-2} \, dx \Rightarrow 3u^2 \, du = dx$$

which means we can in fact perform the substitution:

$$\int \frac{1}{x^{1/3} + 1} \, dx = \frac{1}{u + 1} u^2 \, du.$$ 

OK: much better, but still not solvable yet. But the integrand is just a rational function, and the degree of the numerator is not less than that of the denominator, so let’s just do long division, which yields:

$$\frac{u^2}{u + 1} = u - 1 + \frac{1}{u + 1}.$$ 

Perfect!

$$\frac{1}{u + 1} u^2 \, du = \int (u - 1 + \frac{1}{u + 1}) \, du$$

$$= \frac{1}{2} u^2 - u + \ln |u + 1| + c$$

$$= \frac{1}{2} x^{2/3} - x^{1/3} + \ln |x^{1/3} + 1| + c$$

(where we noticed that the derivative of $\ln |u + 1| = \frac{1}{u+1}$, or else did a little substitution $v = u + 1$, $dv = du$ to figure it out). We check our answer is correct, by differentiating it.

Sometimes, it pays to be persistent and creative with substitutions.

### 3.7.5 Substitutions with trig functions

**Example 3.79.** Find

$$\int \frac{\cos^3(x)}{\sqrt{\sin(x)}} \, dx.$$ 

The most complex term is $\sqrt{\sin(x)}$ so we try

$$u = \sin(x) \quad \Rightarrow \quad du = \cos(x) \, dx.$$ 

But this leaves a $\cos^2(x)$ in the numerator we haven’t been able to remove. So we use the trigonometric identity:

$$\cos^2(x) = 1 - \sin^2(x)$$

and then we have

$$\int \frac{\cos^3(x)}{\sqrt{\sin(x)}} \, dx = \int \frac{1 - \sin^2(x) \cos(x)}{\sqrt{\sin(x)}} \, dx$$

$$= \int \frac{1 - u^2}{\sqrt{u}} \, du$$

$$= \int (u^{-1/2} - u^{3/2}) \, du$$

$$= 2\sqrt{u} - \frac{2}{5} u^{5/2} + c$$

$$= 2\sqrt{\sin(x)} - \frac{2}{5} (\sin(x))^{5/2} + c$$

and we again check by differentiating.
Lesson: don’t forget that you can use trig identities to modify your integrand.

Example 3.80. Find

\[ \int \cos^3(x) \, dx \]

This one doesn’t look like a candidate for substitution at all. We might try

\[ u = \cos(x) \quad \Rightarrow \quad du = -\sin(x) \, dx. \]

AN INCORRECT SUBSTITUTION WOULD BE: \( \int \cos^3(x) \, dx = \int u^3 \, dx. \) THIS IS WRONG: the “integral” on the right has two different variables in it, and makes no sense.

BUT, knowing that if I’m going to do a substitution for one of \( \sin(x) \) or \( \cos(x) \), then I’ll need the other one for the \( du \), I will FIRST use the trig identity \( \cos^2(x) = 1 - \sin^2(x) \) to write

\[ \int \cos^3(x) \, dx = \int (1 - \sin^2(x)) \cos(x) \, dx \]

and then we see that the correct substitution is

\[ u = \sin(x) \quad \Rightarrow \quad du = \cos(x) \, dx \]

which gives

\[
\int (1 - \sin^2(x)) \cos(x) \, dx = \int (1 - u^2) \, du \\
= u - \frac{1}{3} u^3 + c \\
= \sin(x) - \frac{1}{3} \sin^3(x) + c
\]

as we check by differentiation.

Be flexible about your substitutions. Don’t erase one that doesn’t seem to be working, just start fresh (because maybe the “failed” one will suddenly turn out OK once you’ve done some algebra).

3.7.6 Here are some more examples that we did not do in class

Example 3.81. Find

\[ \int \frac{\sin(x)}{1 + \cos^2(x)} \, dx. \]

In this one, the danger is being too greedy with your substitution. If you try \( u = 1 + \cos^2(x) \) then \( du = -2 \cos(x) \sin(x) \, dx \) which is a disaster.

However, if we just try

\[ u = \cos(x) \quad \Rightarrow \quad du = -\sin(x) \, dx \]

then things go very nicely:

\[
\int \frac{\sin(x)}{1 + \cos^2(x)} \, dx = \int \frac{1}{1 + u^2} (-1) \, du \\
= - \int \frac{1}{1 + u^2} \, du \\
= - \arctan(u) + c \\
= - \arctan(\cos(x)) + c
\]

which we check by differentiation.
Example 3.82. Find

\[ \int \frac{3}{1 + 4x^2} \, dx \]

The integrand is almost like an arctangent. We see

\[ \frac{3}{1 + 4x^2} = 3 \left( \frac{1}{1 + (2x)^2} \right) \]

so we try \( u = 2x, \, du = 2\, dx \), which gives

\[ \int \frac{3}{1 + 4x^2} \, dx = \frac{3}{2} \int \frac{1}{1 + u^2} \, du = \frac{3}{2} \arctan(u) + c = \frac{3}{2} \arctan(2x) + c \]

3.7.7 Tips on substitution

- When in doubt, start small. A little linear substitution like \( u = 3x \) or \( u = x - 2 \) is always possible (since \( du = 3\, dx \) or \( du = dx \) can always be done) and it might make the mess look a lot more clear to you.
- Don’t be too greedy with your substitution: don’t take \( u = \ln(\sin(x)) \) in one go but instead start with \( u = \sin(x) \) and see what happens.
- Be meticulous in your work. Sloppy substitution will give you garbage and is worthless.
- Make sure that you have translated every part of your integral to your new variable. Never write any integral with two different variables in it.
- Be flexible. Try a substitution even if you can’t see how it will turn out. If you can’t get it to work, or it gives a yuckier integral, don’t erase it, but try a different substitution.