3.9.3 A partial fractions example

These examples can be very long to do. Please see the textbook for more examples.

Example 3.99. Find

\[ \int \frac{2x^3 + 3x^2 + 22x - 6}{x^3 + 3x^2 + 12x + 10} \, dx \]

Step 1: long division. We compute

\[ \frac{2x^3 + 3x^2 + 22x - 6}{x^3 + 3x^2 + 12x + 10} = 2 - \frac{3x^2 + 2x + 26}{x^3 + 3x^2 + 12x + 10} \]

Step 2: factor. We see that \(-1\) is a root of the denominator, so we divide through by \(x + 1\) to get

\[ x^3 + 3x^2 + 12x + 10 = (x + 1)(x^2 + 2x + 10) = (x + 1)((x + 1)^2 + 9) \]

Step 3: write expanded form of proper fraction. We have

\[ \frac{3x^2 + 2x + 26}{x^3 + 3x^2 + 12x + 10} = \frac{A}{x + 1} + \frac{Bx + C}{(x + 1)^2 + 9} \]

Step 4: solve for coefficients. We cross-multiply to get

\[ 3x^2 + 2x + 26 = A((x + 1)^2 + 9) + (Bx + C)(x + 1) \]

Plug in \(x = -1\) to get \(3 - 2 + 26 = 9A\) or \(A = 3\). Plug in \(x = 0\) to get \(26 = 3(10) + C(1)\) or \(C = -4\). Plug in \(x = 1\), for example, to get \(31 = 3(13) + (B - 4)(2)\) or \(B = 0\). Thus we have

\[ \frac{3x^2 + 2x + 26}{x^3 + 3x^2 + 12x + 10} = \frac{3}{x + 1} + \frac{-4}{(x + 1)^2 + 9} \]

Step 5: integrate. We have

\[ \int \frac{2x^3 + 3x^2 + 22x - 6}{x^3 + 3x^2 + 12x + 10} \, dx = \int \left( 2 - \frac{3x^2 + 2x + 26}{x^3 + 3x^2 + 12x + 10} \right) \, dx \]

\[ = 2x - \int \left( \frac{3}{x + 1} + \frac{-4}{(x + 1)^2 + 9} \right) \, dx \]

\[ = 2x - \int \frac{3}{x + 1} \, dx + 4 \int \frac{1}{(x + 1)^2 + 9} \, dx \]

\[ = 2x - 3 \ln |x + 1| + 4 \int \frac{1}{u^2 + 9} \, du \quad u = x + 1, \, du = dx \]

This last integral can be solved by noting that

\[ \frac{1}{u^2 + 9} = \frac{1}{9((u/3)^2 + 1)} = \frac{1}{9} \frac{1}{(u/3)^2 + 1} \]

so that \(v = u/3, \, dv = \frac{1}{3}du\) or \(du = 3dv\) yields

\[ \int \frac{1}{u^2 + 9} \, du = \frac{1}{9} \int \frac{1}{v^2 + 1} \, dv = \frac{1}{3} \arctan(v) + c = \frac{1}{3} \arctan((x + 1)/3) + c \]

so that our final answer is

\[ 2x - 3 \ln |x + 1| + \frac{4}{3} \arctan((x + 1)/3) + c \]

as we can check by differentiation.
3.10 Lecture 15: Numerical integration

So we have a selection of techniques we can try when we need to integrate a function:

- recognizing the anti-derivative
- substitution
- integration by parts
- trigonometric identities (to integrate products of sine and cosine functions)
- trigonometric substitution (for integrands of a special form)
- partial fractions (for rational functions)

In real life, you also have access to tables of common integrals, so that you don’t need to keep re-deriving the integral of \( \ln(x) \) or other common integrals.

Even so: there are many interesting functions out there which do not admit a nice anti-derivative, like \( e^{-x^2} \) or \( \sqrt{1 + x^3} \). That means you could try all the techniques you like, you’d never get to an integral you could “solve” algebraically.

Remember that the reason we introduced integration was to find the area under the curve; anti-derivatives got involved once we learned the Fundamental Theorem of Calculus, which said that if you have an anti-derivative, then the area problem is very easy. So if we don’t have an anti-derivative, we have to go back to our first principles.

Recall: we can estimate the integral \( \int_a^b f(x) \, dx \) with a Riemann sum of the form

\[
\sum_{i=1}^{n} f(x_i^*) \Delta x
\]

where \( \Delta x = \frac{b-a}{n} \) represented the width of all your rectangles and for each \( i \in \{1, 2, \cdots, n\} \), \( x_i^* \) is some point in the interval \([x_{i-1}, x_i]\), where \( x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = b \).

We often chose the left or right endpoints for \( x_i^* \), but clearly the midpoint would usually be a better choice.

3.10.1 Midpoint rule

For the midpoint rule, calculate the Riemann sum on \( n \) subintervals and choose \( x_i^* = \frac{x_{i-1} + x_i}{2} \) to be the midpoint of the subinterval \([x_{i-1}, x_i]\) each time.

Example 3.100. Estimate the value of \( \int_1^3 e^x \, dx \) using the midpoint rule with 4 subintervals. Then compare with the real value.

Solution: Here we have \( b - a = 2 \) so \( \Delta x = (b - a)/n = 2/4 = 0.5 \). So the endpoints of our subintervals are

\[
1, 1.5, 2, 2.5, 3
\]

and the midpoints of these 4 subintervals are, respectively,

\[
1.25, 1.75, 2.25, 2.75.
\]

Therefore our estimate using the midpoint rule with 4 subintervals is

\[
M_4 = \sum_{i=1}^{n} f \left( \frac{x_{i-1} + x_i}{2} \right) \Delta x \\
= f(1.25)\Delta x + f(1.75)\Delta x + f(2.25)\Delta x + f(2.75)\Delta x \\
= 0.5 \left( f(1.25) + f(1.75) + f(2.25) + f(2.75) \right) \\
\sim 0.5(34.3753) = 17.1877
\]

\(^8\)Of course, if you’re given an integral on an exam and asked to solve it, it is solvable algebraically.
On the other hand,
\[
\int_1^3 e^x \, dx = e^3 \bigg|_1^3 = 17.36725509
\]
so our error was about 0.18.

A problem with the midpoint rule is that it’s awkward to evaluate the function on all these midpoints; and worse, if all you have is a table of numbers, then to use the midpoint rule you need to ignore a lot of values.

**Example 3.101.** Given that \( f(0) = 1, \ f(1) = 2 \) and \( f(2) = 1 \), what’s the best estimate we can make of \( \int_0^2 f(x) \, dx \) using the midpoint rule?

Solution: We can only use one subinterval: \( n = 1 \), so that \( \Delta x = 2 \), and \( M_1 = f(1)\Delta x = 4 \). If we tried using 2 subintervals, we’d need to know \( f(0.5) \) and \( f(1.5) \), which we don’t have.

### 3.10.2 Trapezoidal Rule

Another reasonable approach is to average the answers of the left and right endpoint rules. That is, given the left Riemann sum
\[
L_n = \Delta x \left( f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right)
\]
and the right Riemann sum
\[
R_n = \Delta x \left( f(x_1) + f(x_2) + \cdots + f(x_n) \right)
\]
you take their average, which gives
\[
T_n = \frac{1}{2} (L_n + R_n) = \frac{1}{2} \Delta x \left( f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right)
\]
(where all the inner terms are doubled because they occur in both expressions but the first and last terms only occur in one of the two Riemann sums).

Why do we call this the Trapezoidal rule? Recall that the area of a trapezoid of base \( \Delta x \) and whose heights are \( f(x_{i-1}) \) and \( f(x_i) \) is
\[
\frac{1}{2} \Delta x \left( f(x_{i-1}) + f(x_i) \right)
\]
(or maybe you don’t recall, but we can derive this formula in any case!). So in fact average the left and right Riemann sums corresponds to adding up the areas of all these trapezoids that approximate the curve by linear segments!

**Example 3.102.** Estimate the value of \( \int_1^3 e^x \, dx \) using the midpoint rule with 4 subintervals. Then compare with the real value.

Solution: We have \( \Delta x = 0.5 \) and the endpoints are \( x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3 \).

\[
T_4 = \frac{1}{2} \Delta x \left( f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right)
\]
\[
= \frac{1}{2} (0.5) \left( f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3) \right)
\]
\[
\sim \frac{1}{4} (70.910297) \sim 17.727574
\]
This has an error of 0.36 compared to the true value — it was not as good an estimate as the midpoint rule.

**Remark 3.103.** In fact, one can prove that the error on the trapezoidal rule tends to be about twice that of the midpoint rule, if you use the same number of subintervals; that is, the midpoint rule is better, all things being equal.
More precisely, there exist error bounds: if your function is such that \(|f''(x)| \leq K\) on \([a, b]\), then the errors (difference between estimate and true value) can be proven to be

\[
Err(T_n) \leq \frac{K(b - a)^3}{12n^2} \sim c(\Delta x)^2
\]

(for \(c\) a constant) and

\[
Err(M_n) \leq \frac{K(b - a)^3}{24n^2} \sim c'(\Delta x)^2
\]

(for \(c'\) a constant).

However, if your function is given as a table, then the trapezoidal rule has a distinct advantage, because it can use more subintervals and more endpoints.

**Example 3.104.** Given that \(f(0) = 1\), \(f(1) = 2\) and \(f(2) = 1\), what’s the best estimate we can make of \(\int_0^2 f(x)\,dx\) using the trapezoidal rule?

Solution: We can use two subintervals: \(n = 2\), so that \(\Delta x = 1\), and

\[
T_2 = \frac{1}{2} \Delta x (f(0) + 2f(1) + f(2)) = \frac{1}{2}(1 + 2(2) + 1) = 3.
\]

Looking at our data, and drawing the trapezoids, we feel more confidence in this result than in the midpoint rule.

Another feature of the trapezoidal rule is that it’s often easy to see if your answer is an overestimate or an underestimate. Namely, if the function is concave up, then all the trapezoids will lie above the curve, giving an overestimate of the value of the integral; whereas if the function is concave down, then all the trapezoids lie below the curve, giving an underestimate of the value of the integral. So we knew that \(T_1\) was an overestimate of \(\int_1^3 e^x\,dx\), for example (just from looking at the curve), whereas it would be harder to tell with the midpoint rule.

### 3.10.3 Simpson’s Rule

Our geometric interpretation of the trapezoidal rule (as measuring the area under some kind of piecewise-linear approximation to the curve) leads to the idea: what if we fit quadratic, not just linear, terms to interpolate between the points on the curve? This should give a better fit than trapezoids.

Indeed it does, and the result will be called Simpson’s rule.

How does it work? Suppose we want to estimate \(\int_a^b f(x)\,dx\). We sketch the graph of \(y = f(x)\) and subdivide \([a, b]\) into \(n\) equal subintervals as usual. This time, use three points to fit a quadratic curve.

But how do you fit a quadratic to three points \((x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\)? One way is to write \(q(x) = ax^2 + bx + c\) and solve for \(a, b, c\) by plugging in \(q(x_i) = f(x_i)\) for \(i \in \{0, 1, 2\}\). This gives a system of linear equations you can solve for \(a, b, c\). (See [S, Section 5.9].)

Another way is as follows. Consider the polynomial

\[
q(x) = f(x_0) \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1) \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2) \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}
\]

Remember that \(x_0, x_1, x_2\) are points (numbers); the only variable in this formula is \(x\), and so this is just a quadratic polynomial. Let’s simplify. In our case, the differences \(x_1 - x_0\) and \(x_2 - x_1\) are equal to \(\Delta x\). Let’s write

\[
h = \Delta x,
\]

to make the formula less confusing to look at. So our polynomial is just

\[
q(x) = f(x_0) \frac{1}{2h^2} (x - x_1)(x - x_2) + f(x_1) \frac{1}{-h^2} (x - x_0)(x - x_2) + f(x_2) \frac{1}{2h^2} (x_2 - x_0)(x_2 - x_1).
\]
Now our estimate for \( \int_{x_0}^{x_2} f(x) \; dx \) is the integral
\[
\int_{x_0}^{x_2} q(x) \; dx.
\]
This is clearly the sum of the three integrals above; let’s work one out as the work for the other is similar.

To solve
\[
\int_{x_0}^{x_2} \frac{f(x_0)}{2h^2} (x - x_1)(x - x_2) \; dx
\]
use the substitution: \( t = x - x_1 \), so \( dt = dx \). Then \( x = t + x_1 \) so \( x - x_1 = t \) and \( x - x_2 = t + x_1 - x_2 = t - h \). For the limits: \( x = x_0 \Rightarrow t = -h \), \( x = x_2 \Rightarrow t = h \). This gives
\[
= \frac{f(x_0)}{2h^2} \int_{-h}^{h} t(t - h) dt = \frac{f(x_0)}{2h^2} \int_{-h}^{h} (t^2 - ht) dt.
\]
But \( t^2 \) is even and \( ht \) is odd, so separating this as a sum of two integrals (\( \int_{-h}^{h} (t^2 - ht) dt = \int_{-h}^{h} t^2 dt - h \int_{-h}^{h} t dt \)) we see that the second is zero because the integrand is odd, and the first is just twice the integral from 0 to \( h \) since \( t^2 \) is even. So we have
\[
= 2 \frac{f(x_0)}{2h^2} \int_{0}^{h} t^2 dt = 2 \frac{f(x_0)}{2h^2} \frac{1}{3} h^3 = \frac{1}{3} h f(x_0).
\]
By similar arguments, we get
\[
\int_{x_0}^{x_2} f(x_1) \frac{1}{h^2} (x - x_0)(x - x_2) \; dx = \frac{4}{3} h f(x_1)
\]
and
\[
\int_{x_0}^{x_2} f(x_2) \frac{1}{h^2} (x - x_0)(x - x_1) \; dx = \frac{1}{3} h f(x_2).
\]
We conclude (writing now \( \Delta x = h \) as usual) that
\[
\int_{x_0}^{x_2} q(x) \; dx = \frac{1}{3} \Delta x (f(x_0) + 4f(x_1) + f(x_2)).
\]

Good: so to approximate \( \int_{a}^{b} f(x) \; dx \), we have to add all these integrals together, working in triples, which gives
\[
S_n = \frac{1}{3} \Delta x (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 4f(x_{n-1}) + f(x_n)).
\]
Note that \( n \) had to be even for this to work. The coefficients follow the pattern: 1, 4, 2, 4, 2, 4,...2, 4, 1.

**Example 3.105.** Estimate the value of \( \int_{1}^{3} e^x \; dx \) using Simpson’s rule with 4 subintervals. Then compare with the real value.

We calculated the value \( \Delta x = 0.5 \) and the endpoints above. So we have
\[
S_4 = \frac{1}{3} (0.5)( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4))
= \frac{1}{6} (f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3))
= \frac{1}{6} (104.2386631) \sim 17.37311
\]
This differs from the true value of 17.36725509 by only 0.006. This is a stunningly better approximation!

In general, the error on Simpson’s rule tends to be several orders of magnitude lower than either the midpoint rule or the trapezoidal rule, for the same number of subintervals. One can keep increasing the degree of the curves you use to approximate \( f \), but for everyday purposes Simpson’s rule has won out as the favourite. It is the method programmed into the calculators that you’re not permitted to use on exams in this Faculty, for example.
Remark 3.106. In fact, it can be proven that if $|f^{(4)}(x)| \leq L$ on $[a, b]$ then the error between the correct value and the estimate provided by Simpson’s rule is

$$
Err(S_n) \leq \frac{L(b-a)^5}{180n^4} \sim c(\Delta x)^4
$$

for $c$ some constant.

You can even use Simpson’s on tables of values.

Example 3.107. Given that $f(0) = 1$, $f(1) = 2$ and $f(2) = 1$, what’s the best estimate we can make of $\int_0^2 f(x) \, dx$ using Simpson’s rule?

Solution: We can use two subintervals: $n = 2$, so that $\Delta x = 1$, and

$$
S_2 = \frac{1}{3} \Delta x (f(0) + 4f(1) + f(2)) = \frac{1}{3} (1 + 4(2) + 1) = \frac{10}{3}.
$$

Again, we might feel that this is a better fit, but of course that is subjective since $f$ could be ANY curve passing through those 3 points.