Example 4.13. Find all local extrema of \( f(x) = |3x - 4| \).

We rewrite this as

\[
  f(x) = \begin{cases} 
    3x - 4 & \text{if } 3x - 4 \geq 0 \text{ or } x \geq 4/3 \\
    -(3x - 4) & \text{if } 3x - 4 \leq 0 \text{ or } x \leq 4/3.
  \end{cases}
\]

So the derivative is

\[
  f'(x) = \begin{cases} 
    3 & \text{if } x > 4/3 \\
    -3 & \text{if } x < 4/3.
  \end{cases}
\]

and at \( x = 4/3 \), we see that the function isn’t differentiable. So the only critical point is \( c = 4/3 \). At \( c = 4/3 \), \( f(c) = 0 \), and we see that \( f(x) \geq f(c) \) for every other \( x \), so 0 a local minimum (and in fact an absolute minimum) of \( f \).

4.1.6 Finding absolute extrema

The extreme value theorem tells us that an absolute maximum and minimum of a continuous function of \( f \) on a closed interval \([a, b]\) must exist. Where are they?

If the extremum occurs at an interior point \( c \in (a, b) \) then it is also a local extremum, and so we know that \( c \) is a critical point. Otherwise, \( c \) must have been an endpoint, that is, either \( c = a \) or \( c = b \).

So our method for finding absolute extrema is:

1. Find all critical points of \( f \) on \([a, b]\).
2. Evaluate \( f \) on all critical points and at both endpoints.
3. The largest value is the absolute maximum and the least is the absolute minimum of \( f \) on \([a, b]\).

Example 4.14. Find the absolute max and min of \( f(t) = \frac{4}{3}t - \tan(t) \) on \([-\pi/4, \pi/4]\).

1. The function \( f \) is differentiable everywhere. We find \( f'(t) = \frac{4}{3} - \sec^2(t) \) so \( f'(t) = 0 \) when \( \cos^2(t) = \frac{3}{4} \) or \( \cos(t) = \frac{\sqrt{3}}{2} \). On the interval \([-\pi/4, \pi/4]\), we see that \( \cos(t) = \frac{\sqrt{3}}{2} \) at two points, \( t = \pi/6 \) and \( t = -\pi/6 \).
2. We have

\[
  f(\pi/6) = \frac{4}{3}(\pi/6) - \frac{1}{\sqrt{3}} = 0.121
\]
\[
  f(-\pi/6) = \frac{4}{3}(-\pi/6) - (-\frac{1}{\sqrt{3}}) = -0.121
\]
\[
  f(-\pi/4) = \frac{4}{3}(-\pi/4) - (-1) = -0.0472
\]

and

\[
  f(\pi/4) = \frac{4}{3}(\pi/4) - 1 = 0.0472
\]

3. The absolute max is attained at \( x = \pi/6 \), the absolute min is attained at \( x = -\pi/6 \).

Example 4.15. Same question, but on \([0, \pi/4]\). Since \( f(0) = 0 \) we get our absolute minimum at 0 instead.

4.2 Lecture 17: Limits and continuity of functions

In this lecture, we address the concepts of limits and continuity. See sections 2.2 through 2.5 of the textbook [S], as well as Appendix D. Those of you taking MAT1325 will address this material more thoroughly.

We have been using limits extensively in this course: the concepts of the derivative and the integral fundamental require the notion of a limit, although in different ways; in this chapter we focus on the limit of functions, such as arise when you want to graph \( y = f(x) \), or when you take the derivative.
4.2.1 The definition of the limit, and of continuity

Intuitively: we say \( \lim_{x \to a} f(x) = L \) if the closer \( x \) gets to \( a \), the closer \( f(x) \) gets to \( L \). We want to capture the idea that regardless of what \( f(a) \) actually is (even if it is not defined), the limit exists if the value of the function \( f(x) \) is inexorably drawn to \( L \) as \( x \) gets closer to \( a \). We think of the limit as telling us what all the circumstantial evidence would conclude: looking at the values of \( f(x) \) for \( x \) near \( a \) — but not at \( a \) — it looks like \( f(x) \) is going to take value \( L \) at \( a \).

Newton formulated limits as part of his development of the Calculus, but his definition wasn’t precise enough, and left loopholes that worked as traps into which many of his fellow (less able!) mathematicians fell. It took almost 200 years to develop a sufficient accurate, watertight, definition of “limit”; unsurprisingly, this definition is fairly complex. (See Appendix D.) Essentially, it says that

\[
\lim_{x \to a} f(x) = L
\]

if you can draw a box, of any given height, no matter how small, centered at \((a, L)\) such that — except for maybe \((a, f(a))\), which we ignore — the graph of \( f \) is contained in the box (for that small interval around \( a \)).

Similarly, we can define

\[
\lim_{x \to a^+} f(x) = L
\]

if you can draw such a box so that the graph of \( f \) is contained in the box for \( x > a \) (this is called a right hand limit) and we can define

\[
\lim_{x \to a^-} f(x) = L
\]

if you can draw such a box so that the graph of \( f \) is contained in the box for \( x < a \) (this is called a left hand limit). Clearly if the two limits exist and are equal, that is,

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L
\]

then \( \lim_{x \to a} f(x) = L \).

These conditions can fail in lots of interesting ways.

**Example 4.16.** Consider \( \lim_{x \to 0} \frac{|x|}{x} \).

Since \(|x|\) is defined to be \( x \) if \( x \geq 0 \) and \(-x\) if \( x < 0 \), we should separate the two cases.

For \( x < 0 \), \( \frac{|x|}{x} = \frac{-x}{x} = -1 \). So \( \lim_{x \to 0^-} \frac{|x|}{x} = -1 \).

For \( x > 0 \), \( \frac{|x|}{x} = \frac{x}{x} = 1 \), so \( \lim_{x \to 0^+} \frac{|x|}{x} = 1 \).

Since the two one-sided limits exist but are not equal, the (two-sided) limit doesn’t exist.

Note that it doesn’t matter whether we’d specified a value for the function at \( x = 0 \) because we’re trying to evaluate the limit as \( x \to 0 \), which specifically doesn’t care about what actually happens at \( 0 \).

We can see from the graph that there’s no box of height 0.5, for example, that satisfies our criterion for the limit to exist at \( x = 0 \).

**Example 4.17.** Consider \( f(x) = \sin(\pi/x) \). If you plug in \( x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \), or \( x = 0.1, 0.001, 0.0001, \ldots \), you always get \( f(x) = 0 \), so you might think that \( \lim_{x \to 0} \sin(\pi/x) = 0 \). But this is wrong: if you’d plugged in \( x = 2/\pi, 2/(2\pi), 2/(3\pi), \ldots \), you’d get \( f(x) = 1, -1, 1, -1, \ldots \). Looking at the graph of the function, we see that there’s no box centered on \( x = 0 \) of height 0.5, for example, that can completely contain the graph. That is, it’s not true that \( f \) is inexorably aiming for 0, or any other value, for that matter, even though some sequences seem to make it looks as though it was.

Of course, the most useful functions, and ones we spend most time with, are the ones where the limit agrees with the value of the function at that point.

**Definition 4.18.** A function \( f \) is continuous at a point \( a \) in its domain if \( \lim_{x \to a} f(x) = f(a) \). We say \( f \) is continuous if it is continuous at every point in its domain.
Theorem 4.19. The following functions are continuous:

- polynomials
- rational functions
- exponential functions
- logarithmic functions
- absolute value
- trigonometric functions
- inverse trigonometric functions
- algebraic functions

Now let’s mention the result that we use all the time when we are evaluating derivatives from the definition:

Theorem 4.20. Suppose \( f(x) \) is continuous at \( a \) and \( g(x) \) is a function satisfying

\[
 f(x) = g(x) \quad \text{for all } x \text{ in some interval } (b, a) \text{ with } b < a.
\]

Then \( \lim_{x \to a^-} g(x) \) exists and equals \( f(a) \).

Similarly, if \( g(x) \) is a function satisfying

\[
 f(x) = g(x) \quad \text{for all } x \text{ in some interval } (a, b) \text{ with } b > a.
\]

Then \( \lim_{x \to a^+} g(x) \) exists and equals \( f(a) \).

In particular, if \( g(x) \) coincides with a continuous function everywhere except possibly at \( x = a \), then \( \lim_{x \to a} g(x) \) exists and equals \( f(a) \).

Example 4.21. Consider \( g(x) = \frac{x^2 - 1}{x + 1} \). This is not defined at \( x = 1 \), but whenever \( x \neq 1 \), we have

\[
 g(x) = x + 1 = f(x) \quad \text{for all } x \neq 1
\]

and since \( f(x) \) is just a polynomial, it is continuous. So by the theorem, \( \lim_{x \to 1} g(x) = f(1) = 2 \).

Example 4.22. Is

\[
 f(x) = \begin{cases} 
  e^x & \text{if } x \geq 0 \\
  x + 1 & \text{if } x < 0
\end{cases}
\]

a continuous function? If so, is it differentiable at 0?

Solution: Since \( e^x \) and \( x + 1 \) are continuous, we see that everywhere except possibly at \( x = 0 \) the function \( f(x) \) is continuous. What happens at \( x = 0 \)? By the theorem, it’s easy to take left and right hand limits:

\[
 \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x = e^0 = 1
\]

and

\[
 \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 1) = 0 + 1 = 1
\]

so the left and right limits agree and are equal to each other AND to \( f(0) = 1 \). So \( f \) is continuous everywhere.

The derivative is defined in terms of limits. Again, at every point away from 0, you can find a small interval where \( f(x) \) coincides with a differentiable function, so we know

\[
 f'(x) = \begin{cases} 
  e^x & \text{if } x > 0 \\
  ? & \text{if } x = 0 \\
  1 & \text{if } x < 0.
\end{cases}
\]
Now again, each of the functions \( e^x \) and 1 are continuous for all \( x \), so we just need to check that the left and right limits of \( f'(x) \) agree at \( x = 0 \):

\[
\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} e^x = e^0 = 1
\]

which coincides with the left hand limit, so \( f'(0) \) exists and equals 1 as well.

### 4.2.2 The infinite case

See [S, Chap 2.5].

When the graph of \( f \) has a vertical asymptote at \( x = a \), we write (whichever case applies):

\[
\lim_{x \to a^-} f(x) = \infty \quad \lim_{x \to a^-} f(x) = -\infty
\]

\[
\lim_{x \to a^+} f(x) = \infty \quad \lim_{x \to a^+} f(x) = -\infty
\]

This is a strange (but useful!) convention: the limit doesn’t exist (go check the definition!) but we’re saying that it fails to exist in a certain special way (that the function grows without bound).

Similarly, when the graph of \( f \) has a horizontal asymptote, we write

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L
\]

and this time again our definition of limit doesn’t apply because \( \pm \infty \) are not numbers; this time we mean that as \( x \) grows without bound (in the positive or negative direction, depending), \( f(x) \) approaches \( L \).

Finally, we are also interested in statements like

\[
\lim_{x \to \infty} f(x) = \infty
\]

(with \( \pm \) variations), which tell us in which corner of our graph paper the graph of \( f \) will end up.

There are dangers here as well.

#### Example 4.23.

Let \( a \) and \( b \) be constants. Find

\[
L = \lim_{x \to \infty} \left( \sqrt{x^2 + a} - \sqrt{x^2 + b} \right)
\]

If we “plug in” \( \infty \) for \( x \), we get the expression

\[
\infty - \infty
\]

which is called an “indeterminate form” because there’s no way to make sense of it; we’ll see examples where a limit like this comes out like 1, or 5, or 0 or even \( \infty \). So we have to find another way to write the function so that we can figure out where it’s going.

**Standard clever trick #1: rationalize.**

\[
L = \lim_{x \to \infty} \left( \sqrt{x^2 + a} - \sqrt{x^2 + b} \right) \frac{\sqrt{x^2 + a} + \sqrt{x^2 + b}}{\sqrt{x^2 + a} + \sqrt{x^2 + b}}
\]

\[
= \lim_{x \to \infty} \left( \frac{(x^2 + a) - (x^2 + b)}{\sqrt{x^2 + a} + \sqrt{x^2 + b}} \right)
\]

\[
= \lim_{x \to \infty} \left( \frac{a - b}{\sqrt{x^2 + a} + \sqrt{x^2 + b}} \right)
\]

and now when we plug in \( \infty \) for \( x \), we have \( (a - b)/\infty \) which is shorthand for saying: the denominator grows without bound and the numerator is constant, so as \( x \to \infty \), the fraction goes to 0.
Example 4.24. Find
\[ L = \lim_{x \to \infty} \left( \frac{3x^3 + 2x^2 + 1}{2x^3 - 1} \right). \]

This time when you plug in “\( \infty \)” for \( x \), you get \( \infty / \infty \), which is another indeterminate form.

Standard clever trick #2: divide through by the highest power of \( x \) in the denominator.

(You can also just divide through my the highest power anywhere, and it still works.)

\[
L = \lim_{x \to \infty} \left( \frac{3x^3 + 2x^2 + 1}{2x^3 - 1} \right) = \lim_{x \to \infty} \left( \frac{3 + 2(1/x) + (1/x^3)}{2 - (1/x^3)} \right)
\]

and you see that as \( x \to \infty \), all the terms of the form \( 1/x^n \) vanish and so we’re left with \( 3/2 \), which is the limit.

### 4.2.3 Algebra of limits with infinity

For infinite limits, some arithmetic is valid; for example, let \( c > 0 \) be any finite number then

\[
\begin{align*}
\infty \pm c &= \infty, \quad (\infty)^{-1} = -\infty, \\
\infty + \infty &= \infty, \quad \infty \cdot \infty = \infty, \\
c \cdot \infty &= \infty \\
\frac{c}{0^+} &= \infty, \quad \frac{c}{0^-} = -\infty \\
\frac{c}{\infty} &= 0, \quad \frac{\infty}{c} = \infty \\
\infty / \infty &= \infty \quad \infty / \infty = -\infty \\
0^+ / \infty &= 0, \quad 0^- / \infty = -\infty 
\end{align*}
\]

(along with many other variations).

Example 4.25.

\[
\begin{align*}
\lim_{x \to 3^-} \frac{x + 4}{x - 3} &= \frac{7}{0^-} = -\infty \\
\lim_{x \to \infty} e^{2x} (\frac{1}{x} + 4) &= \infty \cdot 4'' = \infty 
\end{align*}
\]

But the following are called indeterminate forms and their value cannot be assessed without analyzing the functions involved:

- \( \infty - \infty \)
- \( \infty - \infty \)
- \( \infty / \infty \)
- \( 0 / \infty \)
- \( 0 \cdot \infty \)


\[
\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2}
\]

is an indeterminate form of type \( 0/0 \); to find the limit we have to algebraically manipulate the function (here, simplify) to deduce the true value. The point is that the indeterminate form holds no information about the value of the limit.
We’ll discover a beautiful solution for some indeterminate forms in Section 4.5.

4.2.4 For next time: Algebra with limits

Let’s summarize what operations you can perform with limits, up to understanding that certain algebraic operations with $\infty$ are allowed and others are not.

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, where $a, L$ or $M$ could be infinite. Then

- $\lim_{x \to a} (f(x) + g(x)) = L + M$ and $\lim_{x \to a} (f(x) - g(x)) = L - M$ (exception: $\infty - \infty$ is indeterminate)
- $\lim_{x \to a} f(x)g(x) = LM$ (exception: if $L = 0$ and $M = \infty$ then it’s indeterminate)
- $\lim_{x \to a} cf(x) = cL$ for any constant $c$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ iff $M \neq 0$ (exceptions: if $M = 0$ then you might be able to reason out if the left or right hand limits are $\pm\infty$; and if $L = M = \infty$ or $L = M = 0$ then it’s indeterminate.)

Note that this promises that the sum, different, product, quotient and composition of continuous functions is continuous. We use this all the time.

Next: if $h$ is a continuous function then

$$
\lim_{x \to a} h(f(x)) = h \left( \lim_{x \to a} f(x) \right) = h(L).
$$

Example 4.27.

$$
\lim_{x \to (\pi/2)^+} \tan(x) = -\infty \quad \text{whereas} \quad \lim_{x \to (\pi/2)^-} \tan(x) = -\infty
$$

$$
\lim_{x \to \infty} e^x = \infty \quad \text{whereas} \quad \lim_{x \to -\infty} e^x = 0
$$

So therefore

$$
\lim_{x \to \pi/2^+} e^{\tan(x)} = 0
$$

since as $x \to \pi/2^+$, we have $\tan(x) \to -\infty$, so $e^{\tan(x)} \to 0$.

4.2.5 Remark

When limits don’t exist for other reasons (like oscillation), then other things come into play.

Example 4.28.

$$
\lim_{x \to \infty} e^{2x} \cos(x)
$$

does not exist; it varies wildly between large positive and large negative numbers. However,

$$
\lim_{x \to -\infty} e^{-2x} \cos(x) = 0
$$

since the exponential pushes the oscillation to 0.

4.2.6 More about indeterminate forms

Think of simpler cases where plugging in gives $\infty - \infty$, to realize how many different answers there can be:

- $\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$

or

- $\lim_{x \to \infty} ((x + 1) - x) = \lim_{x \to \infty} 1 = 1$

etc.