• \( f'(x) \) is defined on the entire domain of \( f \)
• \( f'(x) = 0 \) when \( 1 = \ln(x) \) or \( x = e \)
• so only one critical point: \( x = e \)
• \( f'(x) > 0 \) if \( x < e \) so increasing there
• \( f'(x) < 0 \) if \( x > e \) so decreasing there
• conclude that there’s a local maximum at \( x = e \): \((e, 1/e)\)

Information from \( f''(x) = \frac{x^2(-1/x)-2x(1-\ln(x))}{x^4} = \frac{-3+2\ln(x)}{x^4} \):

• \( f''(x) \) is defined on the entire domain of \( f \)
• \( f''(x) = 0 \) when \( 3 = 2\ln(x) \) or \( x = e^{3/2} \)
• \( f''(x) < 0 \) when \( x < e^{3/2} \), so concave down there — good, because our critical point in on this interval and we’d said it was a maximum!
• \( f''(x) > 0 \) when \( x > e^{3/2} \), so concave up there
• conclude that we change concavity at \((e^{3/2}, 1.5 * e^{-3/2})\), so this is an inflection point

Putting these clues together gives the graph, and our sketch will be quite accurate.

4.4  Lecture 19: Graphing and linear approximation

4.4.1  Graphing

Let’s do another example of graphing a function using all the information we can readily glean from the function itself, and its first and second derivatives.

Let \( f(x) = e^{1/x} \)

• \( f \) has domain all real numbers except \( x = 0 \).
• \( f \) is always positive.
• Since \( f \) is undefined at 0, we need to find the limit of \( f \) as \( x \) approaches 0, that is, what does \( f \) look like near \( x = 0 \)? So:
  \[
  \lim_{x \to 0^+} e^{1/x} = e^{\infty} = \infty
  \]
  since for \( x < 0, 1/x > 0 \) and as \( x \to 0^+ \) we have \( 1/x \to \infty \); so \( e^{1/x} \to \text{infnty} \) also. On the other hand
  \[
  \lim_{x \to 0^-} e^{1/x} = e^{-\infty} = 0
  \]
  since as \( x \to 0^−, 1/x \to −\infty \); but we know that \( \lim_{x \to -\infty} e^x = 0 \) so we deduce \( e^{1/x} \to 0 \). This is a weird answer; so we check, but it’s right.
• Finally, we would like to know if there are horizontal asymptotes, or more generally, how the graph of \( f \) behaves as \( x \to \infty \) and \( x \to -\infty \):
  \[
  \lim_{x \to \infty} e^{1/x} = e^{1/\infty} = e^0 = 1
  \]
  and
  \[
  \lim_{x \to -\infty} e^{1/x} = e^{-1/\infty} = e^0 = 1
  \]
  so there are horizontal asymptotes at both ends.
Remember: a graph can cross a horizontal asymptote! (Look at our graph for \( y = \ln(x)/x \) from last class, for example.)

Ok, this has given us quite a few details about the graph but now we look for the bumps and valleys, the local extrema, which really start to define the shape of the curve in between the points we’ve figured out so far.

We calculate

\[
f'(x) = -\frac{1}{x^2} e^{1/x}
\]

and then:

- \( f'(x) \) is undefined only at \( x = 0 \), which is not in the domain of \( f \) anyway
- \( f'(x) = 0 \) is never true, since \( f'(x) \) is a product of two functions and neither one is ever zero.
- So \( f \) has no critical point on its domain, meaning it attains no local extrema
- We see that \( e^{1/x} > 0 \) for all \( x \neq 0 \), and \( -1/x^2 < 0 \) for all \( x \neq 0 \). So \( f'(x) < 0 \) for all \( x \neq 0 \).
- This says \( f \) is decreasing on every connected component of its domain. That is, \( f \) is decreasing on \((-\infty, 0)\) and also on \((0, \infty)\).
- It is not correct to say that \( f \) is always decreasing — because in fact it isn’t. We see \( f(-1) = e^{-1} < f(1) = e \). The reason this can happen is because there’s a vertical asymptote in between these two points.

**Remark 4.34.** We might also want to know

\[
\lim_{x \to 0^-} \frac{-1}{x^2} e^{1/x}
\]

since that tells us the angle at which we will be approaching \((0, 0)\). (We don’t need to ask about the horizontal asymptotes (obviously the graph flattens out to horizontal at infinity) or the vertical asymptote (obviously the graph gets steeper and steeper). The above limit is an indeterminate form of type 0/0 so we use l’Hospital’s rule

\[
\lim_{x \to 0^-} \frac{-1}{x^2} e^{1/x} = \lim_{x \to 0^-} \frac{e^{1/x} x^{-2}}{2x} = \lim_{x \to 0^-} \frac{e^{1/x}}{2x^3}
\]

YUCK! This is worse than what we started with!! So let’s flip things around and see if it improves:

\[
\lim_{x \to 0^-} \frac{-1}{x^2} e^{1/x} = \lim_{x \to 0^-} \frac{-x^{-2}}{e^{-1/x}} = \lim_{x \to 0^-} \frac{-x^{-2}}{e^{-1/x}} = \lim_{x \to 0^-} \frac{2x^{-3}}{e^{-1/x}(-x^{-2})} = \frac{2x^{-2}}{e^{-1/x}(-x^{-2})} = \lim_{x \to 0^-} -2e^{1/x} = 0
\]

so the graph comes in to \((0, 0)\) from the left at a shallow angle.

Now for the second derivative:

\[
f''(x) = \frac{x^2(-e^{1/x}(-x^{-2})) - (-e^{1/x})(2x)}{x^4} = e^{1/x} \frac{1 + 2x}{x^4}
\]

- \( f''(x) \) is undefined only for \( x = 0 \).
- \( f''(x) = 0 \) when \( 1 + 2x = 0 \) or \( x = -\frac{1}{2} \)
• For $x < -\frac{1}{2}$, $f''(x) < 0$ so the graph is concave down
• For $-\frac{1}{2} < x < 0$, $f''(x) > 0$ so the graph is concave up
• For $x > 0$, $f''(x) > 0$ so the graph is concave up
• We see that $(-0.5, e^{-2})$ is an inflection point since the graph changes concavity there

From these details, we piece together a very good sketch of the graph.

Graphing is a useful skill! Calculus is about geometry and pictures and tangent lines and graphs; often half of the answer to your question is obvious when you look at the graph.

4.4.2 Linear approximation

This is from [S, Chapter 3.9].

Having the full graph of a function, or a formula for it, is wonderful. But the point of Calculus and the derivative is that locally near any point, the graph of a differentiable function is quite close to its tangent line. In practice, this means you can estimate $f(x)$ for $x$ near $a$ by its linear approximation.

That is: think of the tangent line to $f$ at $a$ as another function (a far simpler function!!!) which is pretty close to $f$ near $a$.

What is a formula for the tangent line to $f$ at $(a, f(a))$?

$$y - f(a) = f'(a)(x - a)$$

or

$$y = f(a) + f'(a)(x - a).$$

So this is a simple function of $x$, which we’ll denote $L(x)$:

$$L(x) = f(a) + f'(a)(x - a)$$

and it is a linear approximation to $f$ in the sense that

• it’s a linear function, and
• $L(x)$ is close to $f(x)$ for $x$ near $a$.

Example 4.35. Find the linear approximation to $f(x) = e^x$ near $x = 0$.

Solution: We write down the equation of the tangent line.

$$L(x) = f(a) + f'(a)(x - a) = e^0 + e^0(x - 0) = 1 + x.$$ 

We compare:

$$f(0.1) = e^{0.1} = 1.10517$$

whereas

$$L(0.1) = 1 + 0.1 = 1.1$$

and we needed a calculator to find $f(0.1)$ whereas we didn’t for $L(0.1)$.

However, if $x$ is not near $a$, then your linear approximation won’t be very good: for example $e^2 = 7.389$ but $L(2) = 1 + 2 = 3$. This is obvious from the graph.
Example 4.36. Find the linear approximation to \( f(x) = \sqrt{5 + x} \) at \( x = 4 \) and use it to approximate \( \sqrt{10} \).

Solution: The point \( a \) is 4 and \( f(4) = \sqrt{9} = 3 \). We have \( f'(x) = \frac{1}{2}(5 + x)^{-1/2} \) so \( f'(4) = \frac{1}{2}(9)^{-1/2} = \frac{1}{6} \). So our linear approximation is

\[
L(x) = f(4) + f'(4)(x - 4) = 3 + \frac{1}{6}(x - 4).
\]

We want to estimate \( \sqrt{10} = f(5) \); since 5 is near 4 we can try the linear approximation at 4, and guess that \( f(5) \sim L(5) \):

\[
L(5) = 3 + \frac{1}{6}(5 - 4) = 3.1667
\]

whereas in fact using our calculators we find \( f(5) = \sqrt{10} = 3.1623 \).

Remark 4.37. In Physics, one routinely approximates \( \sin(x) \) with \( x \) (in radians!) for \( x \) near 0. That’s from \( f(x) = \sin(x) \), \( a = 0 \) having linear approximation

\[
L(x) = \sin(0) + \cos(0)(x - 0) = 0 + 1(x) = x.
\]

Replacing \( \sin(x) \) by \( x \) makes it easier to solve for \( x \) in certain formulas and is accurate enough.

Remark 4.38. In MAT1322/MAT1325, we find even better approximations by incorporating higher derivatives; these are the Taylor polynomials (and series). In MAT2322/MAT2122, we see the higher-dimensional analogues, of approximating surfaces by their tangent planes.

4.4.3 How close is “close”?

A reasonable question is: for what values of \( x \) will my linear approximation be any good?

Example 4.39. Consider the linear approximation to \( 8\sqrt{x} \) near \( x = 16 \). (So \( f'(x) = \frac{8}{2}x^{-1/2} \) which equals 1 at \( x = 16 \) so \( L(x) = 32 + (x - 16) = x + 16 \).)

For what range of \( x \)-values does the estimate \( L(x) \) give \( f(x) \) correct to within 0.1?

Solution: we’re asking when

\[
|L(x) - f(x)| < 0.1
\]

We can draw a graph to better understand the condition we’re asking, and we see that we want to know for which \( x \) we have

\[
L(x) - 0.1 < f(x) < L(x) + 0.1
\]

so that the graph of \( f \) lies in a band around the tangent line. From our graph, and that \( f \) is an increasing function but concave down, and so never goes above its tangent line; the extremes of our interval of accuracy are therefore the intersection points

\[
f(x) = L(x) - 0.1 \\
8\sqrt{x} = (x + 16) - 0.1 = x + 15.9 \\
64x = (x + 15.9)^2 = x^2 + 31.8x + 252.81
\]

\[
x^2 - 32.2x + 252.81 = 0
\]

and by the quadratic formula we get

\[
13.6 \leq x \leq 18.6
\]

For example: \( f(18) = 8\sqrt{18} = 33.94 \) and \( L(18) = 18 + 16 = 34 \).