In practice (like when you’re developing a rule of thumb for drug dosages, for example, or car stopping distances): you choose a convenient $a$, find $L(x)$ for $x$ near $a$, and then graphically, numerically or algebraically determine the range in which this approximation is “valid enough”.

### 4.4.4 Aside: differentials

So the slope of the tangent line is

$$\frac{dy}{dx}.$$ 

We can interpret this as a real fraction if we say that $dx$ represents a small change (what we used to call $h$ or $\Delta x$) in the $x$ direction, and then $dy$ is the corresponding rise (or fall) of the tangent line. That is

$$\frac{dy}{dx} = f'(x)$$

so near $a$,

$$dy = f'(a)dx = f'(a)(x - a)$$

so

$$dy = y - f(a).$$

We can thus express our linear approximation in a way that focuses on how far we are from the original point:

$$f(a) + dy = f(a + dx).$$

### 4.5 Lecture 20: Optimization

We know about extreme values of functions:

- If $f$ is continuous on a closed interval, then $f$ attains both an absolute max and absolute min on that interval (extreme value theorem).
- Local and absolute extrema $(x, f(x))$ can only be found in the following: the endpoints of the interval and the critical points of $f$.

**Example 4.40.** For what number between 0 and 1 is the difference between its square and its cube the greatest?

Solution: “the difference between its square and its cube” means $x^2 - x^3$. “the greatest” implies we’re looking for a max (or at least an extremum). “between 0 and 1” means we’re on the interval $[0, 1]$.

So $f(x) = x^2 - x^3$ implies $f'(x) = 2x - 3x^2$; this is 0 if $x = 0$ or $2 = 3x$ meaning $x = \frac{2}{3}$; both these critical points are indeed between 0 and 1, so we consider them.

Our values are:

$$f(0) = 0, \quad f\left(\frac{2}{3}\right) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}.$$ 

By our theorem, we conclude that the maximum difference occurs at $x = \frac{2}{3}$ and that the maximum difference is $\frac{4}{27}$.

**Example 4.41.** You have 100 metres of fencing with which to construct a pair of congruent rectangular pens sharing one side. What should be the dimensions in order to maximize the total enclosed area?

Solution:

1. **Draw a picture.**

2. **Label anything that seems important.** Let’s label one side (the one across the middle) $x$ and the other side $y$.

3. **Write down all information algebraically.** We are given that the perimeter is 100: $3x + 2y = 100$. 
(4) **Identify the quantity to be optimized.** We want to maximize area.

(5) **Represent this quantity as a function of one variable.** Our area is \( xy \), but this has two variables. But \( 3x + 2y = 100 \) so \( y = 50 - \frac{3}{2}x \) and thus

\[
f(x) = x(50 - \frac{3}{2}x) = 50x - \frac{3}{2}x^2
\]

(6) **Locate the endpoints of the variable (if applicable).** Here, \( x \) is between 0 and 100.

(7) **Find the optimal value (from among endpoints and critical points).** We have \( f'(x) = 50 - 3x \) so the only critical point is \( x = \frac{50}{3} \). Since \( x = 0 \) and \( x = 100 \) both give \( f(x) = 0 \), and \( f\left(\frac{50}{3}\right) > 0 \), this critical point corresponds to the optimal solution. \( A = f\left(\frac{50}{3}\right) \) is the maximal area.

**Example 4.42.** Given a right triangle of height \( h \) and base \( b \), inscribe a rectangle in the triangle as in the picture. How do you maximize the area of the rectangle?

Solution:

(1) **Draw a picture.**

(2) **Label anything that seems important.** Let’s label one side of the rectangle (along the base) \( x \) and the other side \( y \).

(3) **Write down all information algebraically.** The lengths of the sides of the triangle are given.

(4) **Identify the quantity to be optimized.** We want to maximize the area of the rectangle.

(5) **Represent this quantity as a function of one variable.** Our area is \( xy \), but this has two variables. They are related; but how? One option: draw the rectangle at the origin and find the equation of the line which is the hypotenuse; then \( (x, y) \) lies on that line, which gives \( y = -\frac{h}{b}x + h \). Another option: note that the complement of the rectangle is two triangles which are similar to the large triangle. Hence we have

\[
\frac{h}{b} = \frac{h - y}{x} = \frac{y}{b - x};
\]

cross-multiplying the last pair gives \( xy = (b - x)(h - y) = bh - by - hx + xy \) or \( by = bh - hx \) or \( y = h - \frac{h}{b}x \). (Same answer, of course!)

Thus we conclude:

\[
f(x) = xy = x(h - \frac{h}{b}x) = hx - \frac{h}{b}x^2
\]

where the only variable is \( x \); \( h \) and \( b \) are constants.

(6) **Locate the endpoints of the variable (if applicable).** Here, \( x \) is between 0 and \( b \).

(7) **Find the optimal value (from among endpoints and critical points).** We have \( f'(x) = h - 2\frac{h}{b}x \) so the only critical point is \( x = \frac{b}{2} \). Since \( x = 0 \) and \( x = b \) both give \( f(x) = 0 \), and \( f\left(\frac{b}{2}\right) > 0 \), this critical point corresponds to the optimal solution. \( A = f\left(\frac{b}{2}\right) = \frac{hb}{4} \) is the maximal area.

**Example 4.43.** Find the dimensions of the 300 mL soda can which is most cost effective.

Solution: Here, we need to interpret a bit: let’s say cost is proportional to the amount of tin needed, and let’s say that this is in turn proportional to the total surface area of the can.

(1) **Draw a picture.** A cylinder.

(2) **Label anything that seems important.** Top radius \( r \), height \( h \).
Write down all information algebraically. The volume is $\pi r^2 h = 300 \text{ mL} = \text{cm}^3$.

Identify the quantity to be optimized. We want to minimize the surface area.

Represent this quantity as a function of one variable. Surface area is $\pi r^2$ for each of the top and bottom, and $2\pi rh$ for the cylindrical part. Thus

$$A = 2\pi r^2 + 2\pi rh$$

which is a function of two variables; but $h = \frac{300}{\pi r^2}$ so we have our function:

$$f(r) = 2\pi r^2 + 2\pi r \frac{300}{\pi r^2} = 2\pi r^2 + \frac{600}{r}$$

Locate the endpoints of the variable (if applicable). Here, $r$ lies between 0 and $\infty$. (You could probably do better on constraining it, but there are no clear physical limits here, and that’s all you need to worry about for this step.)

Find the optimal value (from among endpoints and critical points). We have

$$f'(r) = 4\pi r - \frac{600}{r^2}$$

which is undefined at $r = 0$ and equals 0 when $4\pi r = 600/r^2$ or $r^3 = 150/\pi$ or $r = \sqrt[3]{150/\pi} \approx 3.63$. Since $\lim_{r\to0^+} f(r) = \infty$ and $\lim_{r\to\infty} f(r) = \infty$, we know the minimum occurs at the critical point $r = 3.63, h = 7.26$.

Units? volume was 300 mL = 300 cc’s so each dimension is cm.

Note that the proportions we got are not so far off of a real soda can.

Example 4.44. Find the point on the parabola $x = 3y^2$ which is closest to the point $(\frac{1}{6}, 3)$.

We sketch the graph; the distance between a point $(x, y)$ on the parabola and $(\frac{1}{6}, 3)$ is given by

$$s = \sqrt{(x - \frac{1}{6})^2 + (y - 3)^2}$$

Although the interval is not closed, we see that definitely there will be a minimum distance. We want to find the minimum value of this distance, but right now there are two variables on the right side; but they are related by $x = 3y^2$, so we simplify

$$s^2 = (3y^2 - \frac{1}{6})^2 + (y - 3)^2$$

and now we can find the minimum of $d$ as a function of $y$. We have

$$2s \frac{ds}{dt} = 2(3y^2 - \frac{1}{6})(6y) + 2(y - 3)$$

or

$$s' = \frac{18y^3 - 3}{s} = \frac{3}{s} g y^3 - 1$$

So $s$ has critical points when $s = 0$ (which does not occur here) or when $9y^3 = 1$ or $y = 9^{-1/3}$. We see the derivative is negative before and positive after the critical point $(3^{-1/3}, 9^{-1/3})$, so this is a minimum. The minimum distance is therefore

$$s = \sqrt{(3^{-1/3} - \frac{1}{6})^2 + (9^{-1/3} - 3)^2} \approx 2.57$$

Example 4.45. Joan is at one side of a circular lake of radius 2 km and wants to reach a point diametrically opposite. She can walk at speed 6 km/h and row a boat at speed 3 km/h. How should she proceed to get there fastest?

We draw a picture.
• If she walks around the lake, that’s a distance of $\pi r = 2\pi$ km so would take $2\pi/6 = \pi/3$ hours.

• If she rows directly across the lake, that’s a distance of $2r = 4$ km so would take $4/3$ hours.

• Or: she could row partway and then walk. Let’s work that out next:

If she rows to a point B past the halfway point then if that point is at angle $\theta$ from the horizontal as measured from the center of the lake, we have that the walking distance is $2\theta$ (in radians, the arc length of a circle of radius 2 and angle $\theta$) and the rowing distance is

$$\sqrt{(2 + 2\cos(\theta))^2 + (2\sin(\theta))^2} = 2\sqrt{2 + 2\cos(\theta)}$$

Therefore the time is

$$t = \frac{2\theta}{6} + \frac{2\sqrt{2 + 2\cos(\theta)}}{3}.$$

We differentiate wrt $\theta$ to get

$$\frac{dt}{d\theta} = \frac{1}{3} - \frac{2}{3} \frac{\sin(\theta)}{\sqrt{2 + 2\cos(\theta)}}$$

whence $\theta = \pi$ is a critical point and $t' = 0$ whenever

$$\sqrt{2 + 2\cos(\theta)} = 2\sin(\theta)$$

which after simplifying and using the identity $\sin^2 \theta = 1 - \cos^2 \theta$ yields the quadratic equation

$$2\cos^2 \theta + \cos(\theta) - 1 = 0$$

whence

$$\cos(\theta) = \frac{1 \pm \sqrt{1 + 8}}{4} \in \{1, -\frac{1}{2}\}.$$

These correspond to

$$\theta \in \{0, 2\pi/3\}.$$

We’ve already considered $\theta = 0$ and the other critical point $\theta = \pi$. When $\theta = 2\pi/3$ we get

$$t = \frac{2\pi}{9} + \frac{2}{3} \approx 1.365h$$

which is longer than either other option. So she should walk around the lake.

**Example 4.46.** What is the largest area you can enclose with one wire of length 1m if you cut it in two pieces and make one into a circle and the other into a square?