Not all exercises will be done in the DGD; but please ensure that you ask the TA to do those exercises that you have difficulty with!

(1) Stewart, App A: # 7, 15, 21, 23, 25, 27, 31, 35, 39

(2) Solve \(-3 < \frac{1}{x} \leq 1\).

(3) Prove the following

(a) \(x < y \iff y - x > 0\).
(b) \(-1 < 0\)
(c) If \(x < y\) and \(a \leq b\) then \(x + a < y + b\).
(d) If \(0 < x < y\) and \(0 < a < b\) then \(ax < by\).
(e) \(a > 0, b > 0 \Rightarrow ab > 0\)
(f) If \(x > 0\) then \(0 < x/2 < x\). Hint: show \(2 > 1\), and then deduce \(1/2 < 1\).

(4) If \(a\) and \(\varepsilon\) are real numbers, prove that the set

\(\{x \mid |x - a| < \varepsilon\}\)

is equal to the interval

\((a - \varepsilon, a + \varepsilon) = \{x \mid a - \varepsilon < x < a + \varepsilon\}\).

(5) If \(a > 0\) and \(|x - a| < \frac{a}{2}\), prove that \(\exists b > 0\) such that \(x > b > 0\).

(6) In each case, find \(a, \varepsilon\) such that \(\{x \mid |x - a| < \varepsilon\}\) is equal to the following intervals.

(a) \((-4, 2)\)
(b) \((0, 1)\)
(c) \((-3, 2)\)
(d) \((n, n + 1)\) where \(n \in \mathbb{N}\)
(e) \((b, c)\) where \(b, c \in \mathbb{R}\)

(7) In each case, write \(\{x \mid |x - a| < \varepsilon\}\) as an interval.

(a) \(a = 4, \varepsilon = 1\)
(b) \(a = -12, \varepsilon = 3\)
(c) \(a = 0.5, \varepsilon = 1\)

(8) Prove that if \(a, b \in \mathbb{R}\) then

\((\forall \varepsilon > 0, |a - b| < \varepsilon) \iff a = b\)

(9) Prove the following, using the triangle inequality

(a) \(\forall x, y \in \mathbb{R}, |x - y| \leq |x| + |y|\)
(b) \(\forall x \in \mathbb{R}, |a - b| \leq |x - a| + |x - b|\)
(c) \(\forall x, y \in \mathbb{R}, |a - b| \leq |y - a| + |x - y| + |x - b|\)
(d) \(\forall x, y \in \mathbb{R}, |x + y| \geq |x| - |y|\).

(10) Let \(a \in \mathbb{R}\). Prove that

\(\left( |a| < \frac{1}{2} \text{ and } |x - a| < 1 \right) \Rightarrow |x| < \frac{3}{2}\)

(11) Prove the following implications.

(a) \(|x| \leq 1 \Rightarrow |x^2 - x - 2| \leq 3|x + 1|\)
(b) \(|x| \leq 1 \Rightarrow |x^2 - 1| \leq 2|x - 1|\)
(c) \(|x - 1| \leq 1 \Rightarrow |x^3 + x - 2| \leq 8|x - 1|\)
(12) Find the supremum and infimum of the following sets:
   (a) Let \( n \in \mathbb{N} \). \( S = \{ \frac{p}{q} \mid p, q \in \mathbb{N}, q \neq 0, p + q = n \} \).
   (b) \( S = \{ x \mid x^2 < 16 \} \).
   (c) \( S = \{ x \mid x^2 < -1 \} \).
   (d) \( S = \{ x \mid (x^2 + 1)^{-1} > \frac{1}{2} \} \).
   (e) \( S = \{ x \mid x \in \mathbb{Z}, x^3 < 7 \} \).
   (f) \( S = \{ x \mid x \in \mathbb{Q}, x^2 < 7 \} \).
   (g) \( S = \{ x \mid |2x + 3| < 7 \} \).

(13) Let \( F \) and \( E \) be two nonempty and bounded sets of real numbers such that \( F \subseteq E \). Show that \( \sup(F) \leq \sup(E) \).

(14) Prove the following for any bounded sets \( S, T \) of real numbers:
   (a) if \( S + T = \{ s + t \mid s \in S, t \in T \} \) then \( \sup(S + T) = \sup(S) + \sup(T) \).
   (b) if \( S \cup T = \{ x \mid x \in S \text{ or } x \in T \} \) (the union of \( S \) and \( T \)) then \( \sup(S \cup T) = \max\{\sup(S), \sup(T)\} \).
   (c) if \( S \cap T = \{ x \mid x \in S \text{ and } x \in T \} \) (the intersection of \( S \) and \( T \)) then \( \sup(S \cap T) \leq \min\{\sup(S), \sup(T)\} \).
      Give an example to show that equality need not hold.

(15) Prove the following statements, using the archimedean property of the real numbers.
   (a) \( \forall \varepsilon > 0, \exists n \in \mathbb{N} \text{ such that } 0 < \frac{1}{n} < \varepsilon \).
   (b) \( \forall x \in \mathbb{R}^+ \text{ and } \forall b \in \mathbb{R} \exists n \in \mathbb{N} \text{ such that } nx > b \).

(16) (If we discussed induction in class) Prove by induction:
   (a) \( \forall n \in \mathbb{N}, \sum_{k=0}^{n} k^3 = \left( \sum_{k=0}^{n} k \right)^2 \).
   (b) \( \forall n \geq 4, n \in \mathbb{N}, n^2 \leq 2^n \) (so here your base case is \( n = 4 \), not \( n = 0 \) or \( n = 1 \)).
   (c) \( \forall n \in \mathbb{N}, \forall a_1, \ldots, a_n \in \mathbb{R} \)
      \[ |a_1 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \]
      (so here your induction is on \( n \), the number of terms you are adding together).