These exercises are intended to help you cement your knowledge and to make concrete the theory we discussed in class. Look over and attempt the exercises before the DGD. Not all exercises will be done in the DGD due to limited time but please ensure that you ask the TA to do those exercises that you had difficulty with!

Questions marked with ** are challenging and interesting, and beyond the scope of this course.

(1) Suppose \( f \) is differentiable on \( \mathbb{R} \). Use the Mean Value theorem to prove that if \( f'(x) = 0 \) for all \( x \in [a, b] \) then \( f \) is constant on that interval.

(2) Prove the following theorem, which is called the Mean Value Theorem for Integrals: If \( f \) is continuous on \([a, b]\) then there is a number \( c \in [a, b] \) such that
\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]
(This expression is the average value of \( f \) on \([a, b]\); see Stewart Chapter 6.5.) Hint: apply the Mean Value Theorem to the function \( F(x) = \int_a^x f(t) \, dt \).

(3) Given the Taylor series expansions of \( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \) (centered at 0, with infinite radius of convergence) and of \( \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \) (centered at 1, with radius of convergence 1), find a Taylor series expansion of each of the following functions. In each case, identify the point at which your Taylor series is centered and give the radius of convergence.

(a) \( \ln(x+3) \)
(b) \( \ln(2x+3) \)
(c) \( e^{x^2} \)
(d) \( \frac{1}{x} \)
(e) \( x e^{x^2} \)
(f) \( x \ln(x) \)

(4) Let \( f(x) = (1+x)^k \) where \( k \in \mathbb{R} \).

(a) Find a general formula for \( f^{(n)}(0) \).
(b) Write down the Taylor series of \( f \) centered at 0. Check your answer with Stewart, page 612; this is called the binomial series.
(c) Use the ratio test (on the absolute values of the terms) to show that this series has radius of convergence equal to 1.
(d) Show that for any fixed \( x_0 \in (-1, 1) \), \( \lim_{n \to \infty} |f^{(n)}(x_0)x_0^n/n!| = 0 \).
(e) Conclude that \( f(x) \) is equal to its Taylor series on this interval.

(5) Let \( f(x) = \arcsin(x) \).

(a) Carefully find its first five derivatives and use these to write down its Taylor polynomials centered at \( a = 0 \): \( T_n(f) \), \( 0 \leq n \leq 5 \).
(b) Find (using methods from Calculus I) the maximum value of the function \( f^{(4)}(x) \) on \([0, 0.9]\).
(c) Use this maximum value and Taylor’s theorem to give an upper bound on \( |R_3(f)| \), for \( x \in [0, 0.9] \). Recall this is the absolute value of the error between \( f(x) \) and \( T_3(f)(x) \).
(d) Compare your answer in (c) with \( |f(0.9) - T_3(f)(0.9)| \). Comment on why they are not equal.

(6) Let \( f(x) = \tan(x) \). Compute the first few terms of its Taylor series centered at 0 in two ways.

(a) Find \( f^{(n)}(0) \) for \( n = 0, 1, 2, 3, 4, 5 \) and write down \( T_5(\tan) \).
(b) Perform long division on the Taylor series of \( \sin(x) \) and \( \cos(x) \):
\[
\sin(x) = x - \frac{1}{6} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots
\]
and
\[
\cos(x) = 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots
\]
(being careful to carry enough terms to ensure that you have accuracy of the coefficients to the fifth degree).

(7) For each of the following, sketch the region, write down the integral that represents the area of that region, and compute the area by evaluating the integral. (Numbers refer to the corresponding question in the textbook by Stewart.)

(a) A square of side length $a$.
(b) $y = e^x$, $y = x^2 - 1$, $x = -1$, $x = 1$ (Ch 6.1 #5)
(c) $x = 1 - y^2$, $x = y^2 - 1$ (Ch 6.1 #9)
(d) $y = 12 - x^2$, $y = x^2 - 6$ (Ch 6.1 #13)
(e) $y = e^x$, $y = xe^x$, $x = 0$ (Ch 6.1 #15)
(f) $y = ex$, $y = xe^x$
(g) $y = \arccos(x)$, $y = \arcsin(x)$, $x = 0$: do this both by integrating with respect to $x$ (need to do by parts) and by integrating with respect to $y$ (easier)
(h) A circle of radius $r$. (Hint: the nicest way to integrate this is with the trigonometric substitution $x = \sin(\theta)$.)

(8) Stewart, Ch 6.1 #30, 31, 41