Branching Rules for
Principal Series Representations
of $GL(3, k)$

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Motivation

- Theory of types
- Relation to orbit method
- Insight into other representations
Notation

$k: p$-adic field, $\text{char}(k) = 0$, residual char $= p > 3$

$\mathcal{R}$: integer ring

$\mathcal{P} = \pi \mathcal{R}$: maximal ideal

$G = \text{GL}(3, k)$

$K = \text{GL}(3, \mathcal{R})$: a maximal compact subgroup

$T, \tilde{T}$: diagonal torus in $G$, in $K$

$B, B$: upper triangular Borel in $G$, in $K$

$\Phi$: roots of $(G, T)$

$\mathcal{U}_a$: root subgroup corresponding to $a \in \Phi$
χ: a (ramified) quasi-character of T

Definition: A principal series representation of G is

\[ V_\chi \doteq \text{Ind}_B^G \chi \]

WLOG: \( \chi|_T = (1, \chi_2, \chi_3) \), with

\[
0 \leq \text{cond}(\chi_2) \leq \text{cond}(\chi_2 \chi_3^{-1}) = \text{cond}(\chi_3)
\]

Goal: Decompose \( \text{Res}_K V_\chi \simeq \text{Ind}_B^K \chi \) into irreducibles.
A first step

\(K_n: \text{nth congruence subgroup of } K\)

Then for \(n \geq N:\)

\[(\text{Ind}^G_B \chi)^{K_n} \simeq \text{Ind}^K_{B K_n} \chi = V_n\]

So

\(V_N \subset V_{N+1} \subset V_{N+2} \subset \ldots \text{Ind}^G_B \chi\)

gives a filtration into \(K\)-invariant subspaces.

**Problem:** \(V_n/V_{n+1}\) is far from irreducible.
A finer filtration

Idea: filtrations of parahoric subgroups like $K$ ⇒ look at the extended building of $G$.

The filtration of $K$ by congruence subgroups $K_n$ ⇔ Moy-Prasad filtration.

Take a concave function: $c: \Phi \to \mathbb{Z}_{\geq 0}$ such that $c(a + b) \leq c(a) + c(b)$. Define the subgroup

$$K_c := T \cdot \prod_{a \in \Phi} U_{a,c(a)}.$$

Set

$$C_c = BK_c.$$
Concretely:

Set

\[ c = (c_1, c_2, c_3) = (c(\alpha_{2,1}), c(\alpha_{3,2}), c(\alpha_{3,1})) \];

then

\[ C_c = \begin{bmatrix} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_1} & \mathcal{R} & \mathcal{R} \\ \mathcal{P}^{c_3} & \mathcal{P}^{c_2} & \mathcal{R} \end{bmatrix} \].

Define the poset

\[ \mathcal{T} \doteq \{(c_1, c_2, c_3) \mid 0 \leq c_1, c_2 \leq c_3 \leq c_1 + c_2 \} \].

\( C_c \) is a subgroup of \( K \iff c \in \mathcal{T} \iff c \) is concave.
Subrepresentations of $V_{\chi}$

Recall $\chi = (1, \chi_2, \chi_3)$; suppose $c \succeq (M, N, N) \doteq m$.

Then $\chi$ extends to a character of $C_c$. Define

$$V_c \doteq \text{Ind}^{K}_{C_c} \tilde{\chi}.$$ 

Then

$$\{V_c \mid c \in \mathcal{T}_m\}$$

is a lattice of $K$-invariant subspaces of $V_{\chi}$, with partial order

$$V_d \subset V_c \iff d \prec c$$
Restatement

New Goal: Analyse the $K$-representations $V_c$.

Wish:

\[
\frac{V_c}{\sum_{d < c} V_d}
\]

were irreducible, but this isn’t always true.
Some results

**Theorem [Campbell-N]:** There exists a complete explicit parametrization of the double cosets in \( C_c \backslash K / C_d \) which support nonzero intertwining operators of \( V_c \) with \( V_d \), for any \( c, d \in T_{\geq m} \).

- defines *distinguished* coset representatives \( R_{c,d} \)
- consistency:

\[
c \preceq c', d \preceq d' \implies R_{c,d} \subseteq R_{c',d'}.
\]
Some consequences

Theorem [Campbell-N]:

- $V_m$ is irreducible
- $(C_m, \chi)$ is a type for $\text{Ind}_\mathcal{B}^G \chi$.

Theorem [Campbell-N]: If $\chi$ and $\tau$ are strongly distinct, with the same central character, then $V_{\chi,c} \nparallel V_{\tau,c}$ for any $c$. 
Example 1

χ : any character for which m = (2, 2, 2)

Consider the decomposition of

\[ V_4 = (\text{Ind}_B^G \chi)^{K_4} \]

We can calculate: dimension of, and (often) the number of irreducibles in, each quotient

\[ V_c \sqcup \sum_{d < c} V_d \]
Example 2

\( \chi : \) any character such that \( \mathbf{m} = (1, 2, 2) \)

Consider the decomposition of \( V_\chi \) obtained from all \( V_\mathbf{c} \), with \( \mathbf{c} \) such that \( c_1 + c_2 + c_3 \leq 9 \).
Future work

• Explicit realization of the irreducibles occurring in the reducible $V_c$ quotients

• More explicit results for unramified principal series

• Relationship with orbits of $K$ on its Lie algebra